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This paper investigates asymptotic and exact lower bounds for the communication costs of distributing Quantum Fourier Transform (QFT) on clique networks of quantum machines. We first show that determining lower/upper bound communication complexity is closely related to the number of machines whose qubit capacity becomes full at different stages of QFT. Subsequently, we provide lower and upper bounds on the number of full machines during the execution of distributed QFT. These bounds help us determine the lower bound of the number of non-local operations that should be performed during distribution of QFT on clique networks. We initially analyze the lower bound on communication complexity for the case where machines have capacity two; i.e., each quantum machine can hold at most two qubits. We show that, in this case, the lower bound is $\Omega(n^2)$, where n denotes the number of input qubits. Considering the existing quadratic upper bound in related work, our lower bound result implies that the existing upper bound is actually tight when machine capacity is two. This is a significant theoretical boundary for developers of distribution compilers. We also show that if machine capacity $c > 2$ is a fixed fraction of n, then the lower bound becomes linear.

1 Introduction

The distribution of Quantum Fourier Transform (QFT) is an important problem as QFT is a crucial building block for many Quantum Algorithms (QAs) (e.g., Shor's factoring algorithm [20]) and its distributed version plays an important role in Distributed Quantum Computing (DQC). Due to limited qubit capacity of Noisy Intermediate Scale Quantum (NISQ) machines, large scale quantum problem solving can be achieved through the distribution of QAs over networks of quantum machines. One of the important challenges in DQC includes the compilation of QAs to their distributed versions based on the constraints of a target Quantum Network (QN). Numerous methods [10, 25, 24] exist for such compilations, most of which strive to minimize the cost of quantum communication during distribution. As such, it is of paramount importance to know the theoretical boundaries of the communication costs when developing compilers. While there are methods that analyze the upper bound communication complexity of the distribution of QFT, there is a need for knowing what the best compilation schemes can achieve in terms of quantum communication costs; i.e., *lower bound*. This paper provides exact and asymptotic lower bounds for the communication complexity of distributing QFT, which identifies the minimum resources needed for distributing QFT.

Most existing methods for the analysis of QFT either provide upper bound asymptotic complexity or focus mainly on the implementation of QFT in hardware rather than its distribution. For example, Kitaev [14] proposed a circuit that approximates QFT for arbitrary dimension N, and proved that the size of such a circuit has an upper bound of $log(N/\epsilon)$ for some error bound ϵ. Coppersmith [5] presents a method for approximating QFT within some error bound ϵ, and shows an upper bound of $O(nlog(n/\epsilon))$ on the size of such an approximate circuit where its dimension is $N = 2^n$ and n denotes the number of qubits. Cleve and Watrous [4] present a parallelized unitary circuit that *approximates* QFT up to some error bound $\epsilon > 0$. They show that, if the dimension of QFT is a power of two, then their circuit size has an upper bound $O(nlog(n/\epsilon))$, which becomes quadratic for exponentially small ϵ. They also give the upper bound $O(n(logn)^2 loglogn)$ on the size of an exact QFT modulo 2^n. Their approach is compositional in a vertical manner in that the quantum circuit is partitioned into sub-circuits that should be composed sequentially, whereas distribution is about horizontal partitioning where qubits are distributed across the network. Yimsiriwattana and Lomonaco [27] present an upper bound $O(n^2)$ for the number of global gates in a network of m machines with capacity k, where n denotes the number of qubits and $n = mk$, but they impose no constraints on network connectivity. A *global gate* is a gate whose inputs are distributed across several machines, thereby requiring quantum communication to make it local. Ferrari *et al.* [10] analyze worst case communication complexity of distribution on a linear nearest-neighbor topology where machine capacity is one. They show that the communication complexity grows quadratically with the number of logical qubits and linearly with the depth of the distributed quantum circuit. In another paper, Ferrari *et al.* [11] present a modular architecture for distribution compilers. Their architecture takes in a quantum circuit and a network configuration, and then generates a compiled circuit ready to be distributed on QN. They show that the overall asymptotic complexity of their approach is $O(n^3)$. Van Meter [22] shows that the number of swap operations for inter-node communication on a linear topology is quadratic in the number of qubits. Yimsiriwattana and Lomonaco [26] show that the asymptotic upper bound of distributing Shor's algorithm, including the QFT circuit, is quadratic in the number of qubits. Escofet *et al.* [9] show that the exact upper bound on the number of global gates is proportional to twice the number of gates with two inputs (i.e., binary gates), and the exact lower bound is proportional to the number of binary gates, which is quadratic in the case of QFT.

This paper investigates the best that distribution schemes can achieve in terms of communication costs of distributing QFT. Specifically, we consider a quantum network with clique topology (i.e., complete graph) where any pair of quantum machines have a direct quantum link between them. The clique topology provides a best case scenario in terms of the costs of establishing quantum link between arbitrary pairs of machines in the network. Then, we present a theorem that shows a remote swap operation becomes unavoidable when a global gate has its input qubits located in distinct full machines. Subsequently, we establish lower and upper bounds on the number of full machines that can exist in each stage of the QFT circuit as it is executed in a distributed fashion. The relation between the number of full machines and the necessity of quantum communication is revealing as it helps us to determine the asymptotic lower bound of distributing QFT on a clique network. We first analyze the lower bound for a clique network of machines with capacity two; i.e., each machine can hold at most two qubits. We show that, in this case, the asymptotic lower bound for communication costs of distributing QFT_n on a clique network is $\Omega(n^2)$, where n denotes the number of input qubits. Then, we generalize this result for the case where machine capacity is $c > 2$, and show that the asymptotic lower bound for communication costs is $\Omega(n^2/c)$. This implies that, if $c = \frac{n}{t}$ where $t > 1$ is a constant independent of the number of machines, then the asymptotic lower bound of the communication costs becomes linear. In other words, in order to achieve linear asymptotic lower bound for communication costs of distributing QFT in a clique network, we need to scale up the machine capacity c with constant ratio of $\frac{n}{t}$ as the number of qubits n increases. If the topology is not clique, then every pair of qubits involved in a global gate in QFT must eventually become local, which will require more remote operations (e.g., swap) to ensure that the pair of input qubits move to the same machine. For example, if the topology is a line/chain of machines, then for every global gate to become local we would need $m/2$ quantum communications on average, where m denotes the number of machines in the network. Thus, whatever lower bound we find for QFT on a clique network, should be multiplied by $m/2$ to give us an average case lower bound on a chain. This shows the significance of our results in that it will enable us to identify lower bounds of quantum communication costs for topologies other than clique too.

Organization. Section 2 provides some preliminary concepts and assumptions used in this work. Then, Section 3 analyzes the communication complexity of distributing QFT over clique networks of two-qubit machines. Section 4 then generalizes the results for networks of c-qubit machines where $c > 2$. Subsequently, Section 5 discusses related works. Finally, Section 6 makes concluding remarks and discusses future work.

2 Preliminaries

This section provides some basic concepts and assumptions required for the complexity analysis in this paper. Subsection 2.1 covers quantum circuits and their distribution at a high level of abstraction. Then, Subsection 2.2 represents the Quantum Fourier Transform (QFT) circuit that we analyze for distribution over a clique network. Finally, Subsection 2.3 introduces some of the underlying assumptions we make in our analysis.

2.1 Quantum Circuits and Their Distribution

Quantum Circuits (QCs) capture the logic and order of gate-based quantum transformations that are performed on quantum information bits (i.e., *qubits*) towards solving a problem. A circuit contains a set of horizontal wires carrying quantum information from left to right and quantum gates applied on a subset of wires vertically [17]. There is a one-to-one correspondence between the input qubits of a circuit and its wires. For example, Figure 1-(a) illustrates a quantum circuit processing four qubits through applying a set of gates. A quantum machine runs a circuit by executing its gates from left to right. Notice that, some gates have a single qubit as their input and some take two qubits (a.k.a. *binary gates*) in Figure 1-(a). In general, gates might have multiple input qubits; however, it is known [15] that any quantum circuit can be represented by another circuit formed of a universal set of single-qubit and binary gates to some degree of accuracy. In this paper, we consider an abstract representation of quantum circuits, called the *circuit graph* (e.g., see Figure1-(b)), where we consider a vertex as a point of intersection between a wire and a gate, and an edge represents a binary gate. Each *layer* in a circuit is a collection of gates whose input qubits are disjoint and can be executed simultaneously; e.g., the first two binary gates in Figure 1-(a) form the first layer of the circuit.

Circuit distribution is performed horizontally where proper subsets of qubits are assigned to different quantum machines in the network (see Figure 2). As a result, the inputs of some gates may be located in different machines, called *global gates*; e.g., the first H gate from left in Figure 2. To enable the execution of such gates, they should be localized. That is, make the remote inputs locally available to the machine that is supposed to execute a global gate g, thereby turning g into a *local gate*. We use the term 'localization' for both global gates and their input qubits. Such *localization* requires a means for the transmission of quantum information from one machine to another. However, quantum information cannot be copied [23], nor can it be communicated without error. There are reliable primitives for communicating quantum information,

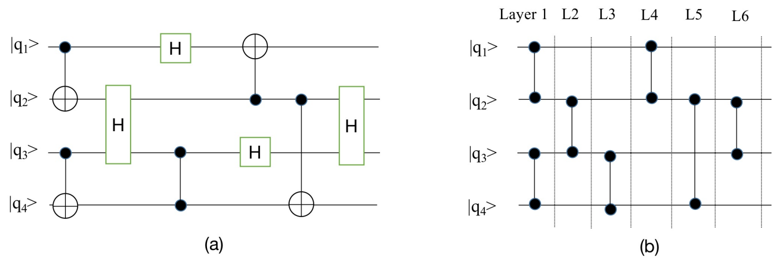

Figure 1: Abstracting circuit (a) as the circuit graph (b).

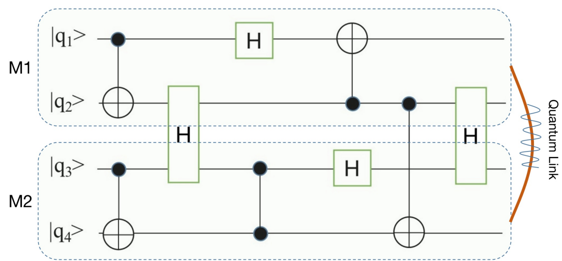

Figure 2: A distributed circuit on two machines M1 and M2 connected by a quantum link..

including *teleportation* [3] (a.k.a. TeleData) and *Cat-entanglement/disentanglement* (Cat-Ent/Cat-DisEnt) [8] (a.k.a. TeleGate). Without loss of generality (Wlog), we use the number of teleportations as an indicator of the number of occasions where such localizations are required, but our analysis is independent of what kind of remote operation is used. For example, to teleport a qubit of information from a location loc_0 to another location loc_1, we need two classic bits of information as well as an Entangled Pair (EP) of qubits, called *Ebit*, that are already distributed over loc_0 and loc_1. The teleportation of a qubit from loc_0 to loc_1 can then be achieved through the execution of some local quantum gates in loc_0 and loc_1. After teleportation, the Ebit is consumed and another one should be generated next time locations loc_0 and loc_1 want to communicate; i.e., *link entanglement generation*. Likewise, the execution of Cat-Ent consumes an Ebit. Another common quantum communication primitive includes the *swap* operation between two qubits q_1 and q_2 respectively located on two distinct machines M_1 and M_2. Such a swap operation can be implemented using two teleportations; one for teleporting q_1 to M_2 and another for teleporting q_2 to M_1. Teleportation is a destructive operation in that when a qubit is teleported to a destination machine, it will be destroyed in the source machine. We ignore the technical details of how such communications take place as our objective is to analyze the number of times such primitives are required for localization.

2.2 Quantum Fourier Transform (QFT)

QFT is a crucial building block of many important QAs such as Shor's factoring and discrete logarithm algorithms [20] and quantum phase estimation [14]. While there are different methods [14, 5, 4, 2] for the implementation of the QFT circuit in hardware, our focus is on its distribution on a network. Implementation and distribution of quantum circuits are similar in that both approaches partition the circuit towards an efficient implementation/distribution. However, the two problems have different constraints and differ in the way they partition a circuit (e.g., vertical vs. horizontal partitioning). In this paper, we focus on the distribution of a standard unitary gate-based QFT circuit using Hadamard and rotation gates. Figure 3 illustrates the QFT_5 circuit and a generalized QFT_n circuit is shown in Figure 4, where n denotes the number of qubits. Since single-qubit gates have no role in making global gates, we eliminate them to get the circuit in Figure 4, which contains only binary rotation gates. For simplicity, we analyze the distribution of

QFT based on its flow of execution from left to right and consider its sub-circuit, separated by dashed red lines in Figure 4. For example, the front sub-circuit of QFT_n includes the rotation gates $R_2, \cdots, R_n$, then the front sub-circuit of QFT_{n-1} has the rotation gates $R_2, \cdots, R_{n-1}$, and so on. Observe that, QFT has a recursive structure where lower size QFT circuits are used in building larger size QFT circuits. For instance, QFT_{n-1} is used to construct QFT_n by adding the front sub-circuit of QFT_n. Next, we present some lemmas on the structure of QFT_n.

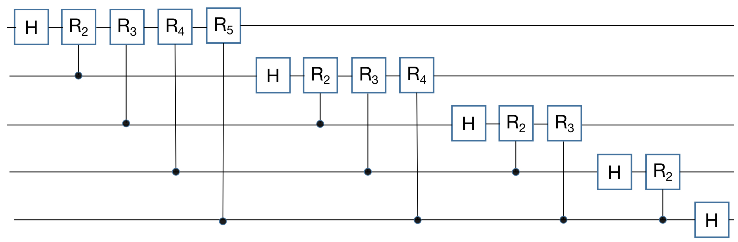

Figure 3: The QFT_5 circuit.

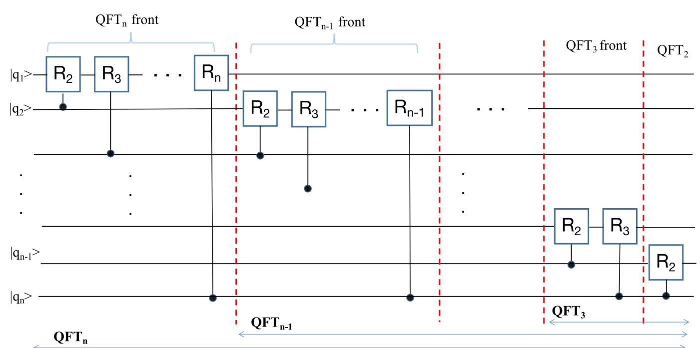

Figure 4: The sub-circuits of QFT_n after removing the single-qubit gates.

Lemma 1. *Let $\mathcal{C}$ be the revised QFT_n circuit that excludes the single-qubit Hadamard gates and includes only the two-qubit rotation gates $R_2, \cdots, R_n$. $\mathcal{C}$ has $\frac{n(n-1)}{2}$ layers.*

Proof. We can partition $\mathcal{C}$ into sub-circuits as in Figure 4. This way, QFT_2 has just one layer. The QFT_3 front circuit has two layers, QFT_4 front has three layers and QFT_5 front has four layers. Thus, QFT_n front has $n-1$ layers. Adding up the number of layers of sub-cricuits, we get the summation of the arithmetic series $1 + 2 + \cdots + (n-1) = \frac{n(n-1)}{2}$. □

Lemma 2. *The i-th front sub-circuit of QFT_n (from left to right) has $n-i$ layers, where $1 \leq i \leq n-2$.*

Proof. We show this lemma by induction on i.
Base Case: Let $i = 1$. Observe that, the first front sub-circuit of QFT_n has gates $R_2, \cdots, R_n$, where each gate forms a layer. Thus, we have $n-1$ layers.
Induction Hypothesis: The i-th front sub-circuit of QFT_n has $n-i$ layers.
Inductive Step: We show that the $(i+1)$-th front sub-circuit of QFT_n has $n-(i+1)$ layers. Once the execution of the i-th front sub-circuit is finished, the remaining gates/layers will be one less than the i-th front sub-circuit, which would be $n-i-1 = n-(i+1)$ layers. □

2.3 Assumptions

We make the following assumptions in this work:

1. Moving from a front sub-circuit of QFT with k qubits to the subsequent sub-circuit with $k-1$ qubits (see Figure 4), we assume that the first qubit can be measured based on the principle of deferred measurement. The rationale behind this assumption is that such measurements free some space in some quantum machines and help us analyze the best case scenario.

2. We consider an abstract unit of cost, which captures the need for the localization of a rotation gate. Our complexity analysis is based on the number of such localization and the implementation of such localizations (e.g., teleportation vs. Cat-Ent/DisEnt) is irrelevant to our analysis.

3. We assume that each quantum machine has an internal clique topology between its qubits; i.e., any local binary gate can be executed on any pair of qubits at no cost. This is a best case assumption in order to analyze the best lower bound communication complexity we can get for the distribution of QFT.

4. A swap operation costs two teleportations.

5. We assume that every individual machine has all the rotation gates $R_2, \cdots, R_n$ used in QFT_n.

3 Lower Bounds for Networks of Two-Qubit Machines

This section investigates the costs of quantum communication when QFT_n is distributed on a clique network of quantum machines with *capacity two*. Even if machines have larger capacities, smaller number of qubits might be allocated for QFT while executing an algorithm that invokes QFT. In the following lemmas, let q_f be the first qubit in the i-th front sub-circuit of QFT_n (which goes through the gates R_2 to R_{n+1-i} for $1 \leq i \leq n-2$), and q_t be the target qubit that should be localized with q_f in order to enable the local execution of some R_j gate, where $2 \leq j \leq n$. Moreover, $m = \lceil \frac{n}{2} \rceil$ denotes the number of machines in the quantum network. Let M_f and M_t be the machines that respectively hold q_f and q_t. We state the following lemma:

Lemma 3. *The cost of a swap is unavoidable iff both M_f and M_t are full where $M_f \neq M_t$.*

Proof. Proof of $\Rightarrow$: We prove the contrapositive of this part. That is, if M_f is not full or M_t is not full, then a swap can be avoided. Trivially in this case we can teleport either q_f to M_t or q_t to M_f; hence no swaps needed.
Proof of $\Leftarrow$: Let q'_f and q'_t be the qubits that are co-located with q_f and q_t respectively in M_f and M_t. If M_f is full and M_t is also full where $M_t \neq M_f$, then there are three ways to localize q_f and q_t: (1) teleport q_t to an empty machine M_e, and then teleport q_f to M_e too; (2) swap q_f with q'_t, or (3) swap q_t with q'_f. The first case takes two teleportation operations, which is the same as the cost of a swap. The other two cases each take a swap. Thus, the cost of a swap is unavoidable. □

Remark. Based on the principle of deferred measurement, q_f can be measured once the execution of the current (i.e., i-th) front circuit is finished. That is, the first top qubit in the i-th front sub-circuit of QFT_n can be measured. For example, in the QFT_5 front sub-circuit of Figure 3, q_1 can be measured after the first four layers are executed. Once a qubit is measured, the machine that holds it will free up space for one qubit.

Lemma 4. *The number of remaining (i.e., unmeasured) qubits during the execution of the i-th front sub-circuit of QFT_n is $(n+1-i)$, where $1 \leq i \leq n-2$. That is, after the execution of the i-th front sub-circuit, the number of remaining qubits is $n-i$.*

Proof. We show this lemma by induction on i.

Base Case: Let $i = 1$. Thus, we have the first front sub-circuit of QFT_n with n qubits, which can be derived from $n+1-1$. After the first front sub-circuit is executed, the top qubit can be measured and we are left with $n-1$ qubits.

Induction Hypothesis: The number of remaining (i.e., unmeasured) qubits during the execution of the i-th front sub-circuit of QFT_n is $(n+1-i)$. After the execution of the i-th front sub-circuit, the number of remaining qubits is $n-i$.

Inductive Step: We show that the number of remaining (i.e., unmeasured) qubits in the $(i+1)$-th front sub-circuit of QFT_n is $(n+1-(i+1))$. Based on the hypothesis, at the start of the $(i+1)$-th front sub-circuit we have $(n-i)$ qubits, which is equal to $(n+1)-(i+1)$. After the execution of the $(i+1)$-th front sub-circuit, the top qubit can be measured, and we would have $((n-i)-1) = n-(i+1)$ remaining qubits. □

Lemma 5. *Let $n > 2$ be an even value. After the i-th front sub-circuit is executed in QFT_n, where $1 \leq i \leq \frac{n}{2}$, the minimum number of full machines in the quantum network is $(m-i)$ and the maximum number of full machines is $m - \lceil \frac{i}{2} \rceil$.*

Proof. We first prove this lemma by an induction on n for a fixed i. Then, we assume that n is fixed and prove the lemma by induction on i.
Base Case: We start the base case with $n = 4$ because QFT_2 is not a distributed circuit. As such, $m = \frac{n}{2} = 2$ and $1 \leq i \leq 2$. After the QFT_4 front circuit (see Figure 3) is executed (i.e., $i = 1$), the top qubit can be measured and we have one half-full machine. Thus, one machine is full and the other is half full. For $i = 1$, we have $m - \lceil \frac{i}{2} \rceil = 2 - 1 = 1$, which is the maximum number of full machines, and $m - 1 = 2 - 1$ as the minimum number of full machines. When QFT_3 front circuit (see Figure 3) is executed (i.e., $i = 2$), we have $m - \lceil \frac{i}{2} \rceil = 2 - 1 = 1$ as the maximum number of full machines, and $m - 2 = 2 - 2 = 0$ as the minimum number of full machines. The maximum case occurs when both measure qubits are in one machine and the minimum occurs when one qubit from each machine is measured. For the case of $n = 6$, we have $m = 3, 1 \leq i \leq 3$. If $i = 1$, then we have at least $m - i = 3 - 1 = 2$ full machines (because we are left with five qubits and three machines), and at most $= m - \lceil \frac{1}{2} \rceil = 3 - 1 = 2$ full machines. When $i = 2$, there is at least $m - i = 3 - 2 = 1$ full machine (i.e., four qubits and three machines, where at least one machine must be full), and at most $m - \lceil \frac{2}{2} \rceil = 3 - 1 = 2$ full machines (when one machine is vacant and four qubits are distributed over two machines). Finally, for $i = 3$, we have at least $m - i = 3 - 3 = 0$ full machine (when each machine holds one of the remaining three qubits), and at most $m - \lceil \frac{3}{2} \rceil = 3 - 2 = 1$ full machines.

Induction Hypothesis: For even values of n, after the i-th front sub-circuit is executed in QFT_n, where $1 \leq i \leq \frac{n}{2}$, the minimum number of full machines in the quantum network is $(m-i)$ and the maximum number of full machines is $m - \lceil \frac{i}{2} \rceil$.

Inductive Step: We show that, after the i-th front sub-circuit is executed in QFT_{n+2}, where $1 \leq i \leq \frac{n+2}{2}$, there are at most $m - \lceil \frac{i}{2} \rceil$ full machines and at least $(m-i)$ full machines in the quantum network, where $m = \frac{n+2}{2}$. We start from the first front sub-circuit ($i = 1$). After executing the QFT_{n+2} front sub-circuit and measuring the top qubit, we have one machine that has only one qubit, and the remaining $m-1$ machines remain full. Thus, we have at least $m - 1 = m - i$ full machines and at most $m - \lceil \frac{i}{2} \rceil = m - \lceil \frac{1}{2} \rceil = m - 1$ full machines. Now, let $i = 2$. At this step, the QFT_{n+1} front sub-circuit is executed and the second qubit is measured. Measuring the second qubit may result in another half-full machines or an empty machine. In the former case, $m-2$ machines would be full, and in the latter case, we have $m-1$ full machines and a vacant machine. This matches with the statement of the lemma that we have at least $m - i = m - 2$ full machines and at most $m - \lceil \frac{i}{2} \rceil = m - \lceil \frac{2}{2} \rceil = m - 1$ full machines. After measuring two qubits, what is left is QFT_n. Based on the induction hypothesis, the statement of the lemma holds for QFT_n. Therefore, the lemma holds for QFT_{n+2} too.

Induction on i. We now prove the lemma by induction on i for a fixed even n.

Base Case: Let $i = 1$ and n be a fixed even value. Observe that, since n is even, $m = \frac{n}{2}$ and all machines are initially full. After the first front sub-circuit is executed in QFT_n, we can measure the top qubit q_f, and we are left with a half-full machine and $m-1$ full machines; i.e.; $m - i$ and $m - \lceil \frac{i}{2} \rceil$ are both equal to $m-1$ when $i = 1$.

Induction Hypothesis: For even values of n, after the i-th front sub-circuit is executed in QFT_n, where $1 \leq i \leq \frac{n}{2}$, the minimum number of full machines in the quantum network is $(m-i)$ and the maximum number of full machines is $m - \lceil \frac{i}{2} \rceil$.

Inductive Step: We show that after the $(i+1)$-th front sub-circuit is executed in QFT_n, where $1 \leq i < \frac{n}{2}$, the minimum number of full machines in the quantum network is $m-(i+1)$ and the maximum number of full machines is $m - \lceil \frac{i+1}{2} \rceil$. By induction hypothesis, the minimum and maximum number of full machines after the i-th front sub-circuit is executed are respectively $(m-i)$ and $m - \lceil \frac{i}{2} \rceil$. Based on Lemma 4, we have $n-i$ qubits at this point. After the execution of the $(i+1)$-th front sub-circuit, the top qubit can be measured and we have one less qubit; i.e., $n-i-1$ remaining qubits. Considering the case of qubit distribution where we have at least $m-i$ full machines in the i-th front sub-circuit, we now have one less full machine since there is one less qubit. Thus, the minimum number of full machines over any qubit distribution is $m-i-1$, which is equal to $m-(i+1)$. Moreover, the maximum number of full machines occurs when all the measured qubits form the minimum number of non-full machines. Thus, the number of remaining full machines could be at most $m - \lceil \frac{i+1}{2} \rceil$ where $\lfloor \frac{i+1}{2} \rfloor$ captures the minimum number of vacant machines. Since there might also be a half-full machine, we get $\lceil \frac{i+1}{2} \rceil$ as the minimum number of non-full machines, thereby maximizing $m - \lceil \frac{i+1}{2} \rceil$. □

Lemma 6. *Let $n > 2$ be an odd value. After the i-th front sub-circuits is executed in QFT_n, where $1 \leq i \leq \lfloor \frac{n}{2} \rfloor$, the minimum number of full machines in the quantum network is $m-i-1$, and the maximum number of full machines is equal to $m - \lceil \frac{i+1}{2} \rceil$.*

Proof. We prove this lemma by an induction on n for a fixed i, and then an induction on i for a fixed n.
Base Case: We start the base case with $n = 3$, where $m = \lceil \frac{n}{2} \rceil = 2$. The QFT_3 has just one front sub-circuit composed with QFT_2. Thus, when $i = 1$ we execute the QFT_3 front sub-circuit (see Figure 3) and measure the top qubit. Since we start with two machines and three qubits, one of the machines is initially half-full. After measuring the first qubit, we either have a vacant machine and a full machine, or two half-full machines. Validating the statement of the lemma, we have at least $m-i-1 = 0$ full machines and at most $m - \lceil \frac{i+1}{2} \rceil = 2-1 = 1$ full machines. For the case of $n = 5$, we have $m = 3, 1 \leq i \leq 2$. Thus, when $i = 1$, we have at least $m-i-1 = 3-1-1 = 1$ full machine, and at most $m - \lceil \frac{i+1}{2} \rceil = 3-1 = 2$ full machines. Note that, for QFT_5, we start with two full machines and a half-full machine. Thus, once the first qubit is measured, we may have either two half-full machines and a full machine, or a vacant machine and two full machines. After measuring the second qubit (i.e., $i = 2$), we have three remaining qubits and three machines. Thus, we have at least $m-i-1 = 3-2-1 = 0$ full machine, and at most $m - \lceil \frac{i+1}{2} \rceil = 3-2 = 1$ full machine.

Induction Hypothesis: For odd values of n, after the i-th front sub-circuits is executed in QFT_n, where $1 \leq i \leq \lfloor \frac{n}{2} \rfloor$, the minimum number of full machines in the quantum network is $m-i-1$, and the maximum number of full machines is equal to $m - \lceil \frac{i+1}{2} \rceil$.

Inductive Step: We start from the first front sub-circuit of QFT_{n+2} ($i = 1$). Note that, we have $m = \lceil \frac{n+2}{2} \rceil$ machines in the network, where one of them is half full. After executing the QFT_{n+2} front sub-circuit and measuring the top qubit, we may have two machines each having only one qubit, and the remaining $m-2$ machines are full, or we have $m-1$ full machines and an empty machine. Thus, we have at least $m-i-1 = m-1-1 = m-2$ full machines and at most $m - \lceil \frac{i+1}{2} \rceil = m - \lceil \frac{2}{2} \rceil = m-1$ full machines, which is consistent with the previous sentence.

Now, let $i = 2$, where one qubit has already been measured and either a machine is empty (i.e., $m-1$ full machines) or there are two half-full machines (i.e., $m-2$ full machines). At this step, the QFT_{n+1} front sub-circuit is executed and the second qubit is measured. Measuring the second qubit may result in two cases: (1) a half-full machine, an empty machine and $m-2$ full machines, or (2) three half-full machines and $m-3$ full machines. This matches with the statement of the lemma that we have at least $m-i-1 = m-2-1 = m-3$ full machines and at most $m - \lceil \frac{i+1}{2} \rceil = m - \lceil \frac{3}{2} \rceil = m-2$ full machines. After measuring two qubits, what is left is QFT_n. The statement of the lemma holds for QFT_n based on the induction hypothesis.

Induction on i. We now prove the lemma by induction on i for a fixed odd n.

Base Case: Let $i = 1$ and n be a fixed odd value. Observe that, since n is odd, $m = \lceil \frac{n}{2} \rceil = \frac{n+1}{2}$. Initially, $m-1$ machines are full and there is a half-full machine. After the first front sub-circuit is executed in QFT_n, we can measure the top qubit q_f, and we are left with either a vacant machine and $m-1$ full machines, or two half-full machines and $m-2$ full machines. Thus, the minimum number of full machines is $m-1-1=m-2$ and the maximum number of full machines is $m-\lceil\frac{1+1}{2}\rceil=m-1$ when $i=1$.

Induction Hypothesis: For odd values of n, after the i-th front sub-circuit is executed in QFT_n, where $1\le i\le\lfloor\frac{n}{2}\rfloor$, the minimum number of full machines in the quantum network is $m-i-1$ and the maximum number of full machines is $m-\lceil\frac{i+1}{2}\rceil$.

Inductive Step: We show that after the $(i+1)$-th front sub-circuit is executed in QFT_n, where $1\le i<\lfloor\frac{n}{2}\rfloor$, the minimum number of full machines in the quantum network is $m-(i+1)-1$ and the maximum number of full machines is $m-\lceil\frac{(i+1)+1}{2}\rceil$. By induction hypothesis, the minimum and maximum number of full machines after the i-th front sub-circuit is executed are respectively $(m-i)-1$ and $m-\lceil\frac{i+1}{2}\rceil$. Based on Lemma 4, we have $n-i$ qubits at this point. After the execution of the $(i+1)$-th front sub-circuit, the top qubit can be measured and we have one less qubit; i.e., $n-i-1$ remaining qubits. Considering the case of qubit distribution where we have at least $m-i-1$ full machines in the i-th front sub-circuit, we now have one less full machine since there is one less qubit. Thus, the minimum number of full machines over any qubit distribution is $m-i-1-1$, which is equal to $m-(i+1)-1$. Moreover, the maximum number of full machines occurs when all the measured qubits form the minimum number of non-full machines. Thus, the number of remaining full machines could be at most $m-\lceil\frac{(i+1)+1}{2}\rceil$ where $\lfloor\frac{(i+1)+1}{2}\rfloor$ captures the minimum number of vacant machines. Since there is also a half-full machine, we get $\lceil\frac{(i+1)+1}{2}\rceil$ as the minimum number of non-full machines, thereby giving us maximum $m-\lceil\frac{(i+1)+1}{2}\rceil$ full machines. □

Corollary 1. *After the i-th front sub-circuits is executed in QFT_n, where $1\le i\le\lfloor\frac{n}{2}\rfloor$, there can be no less than $m-i-1$ full machines in the quantum network. (Proof follows from Lemmas 5 and 6.)*

Lemma 7. *After the $\lfloor\frac{n}{2}\rfloor$-th front sub-circuit is executed in QFT_n, the minimum and maximum number of full machines in the quantum network are respectively zero and $\lceil\frac{(n-\lfloor\frac{n}{2}\rfloor-i)}{2}\rceil$, for $1\le i\le(n-2)-\lfloor\frac{n}{2}\rfloor$.*

Proof. We first prove this lemma by induction on n, and then perform an induction on i assuming n is fixed. *Base Case*: We show the base case for $n=5,6,7$ because $(n-2)-\lfloor\frac{n}{2}\rfloor=0$ for $n=3,4$. Let $n=5$. Thus, we have $(5-2)-\lfloor\frac{5}{2}\rfloor=1$, $m=3$ and $1\le i\le1$. After the second (i.e., $\lfloor\frac{5}{2}\rfloor=2$) front sub-circuit is executed in QFT_5, two qubits have been measured and we are left with three qubits and three machines. The three qubits could each be in a machine, hence leaving us no full machines. Alternatively, two of the qubits may be in one machine, giving us one full machine, which is consistent with $\lceil\frac{(n-\lfloor\frac{n}{2}\rfloor-i)}{2}\rceil=1$ for $n=5$ and $i=1$. If $n=6$ (for an even value), then we have $m=3$, $(6-2)-\lfloor\frac{6}{2}\rfloor=1$ and $1\le i\le1$. After the third (i.e., $\lfloor\frac{6}{2}\rfloor=3$) front sub-circuit is executed (i.e., QFT_6, QFT_5, QFT_4 front sub-circuits), we have three machines and three qubits, which can give us at least 0 and at most 1 full machine (based on a reasoning similar to the case of $n=5$). This validates $\lceil\frac{(n-\lfloor\frac{n}{2}\rfloor-i)}{2}\rceil=1$ for $n=6$ and $i=1$. If $n=7$, then we have $m=4$, $(7-2)-\lfloor\frac{7}{2}\rfloor=2$ and $1\le i\le2$. After the third (i.e., $\lfloor\frac{7}{2}\rfloor=3$) front sub-circuit is executed (i.e., QFT_7, QFT_6, QFT_5 front sub-circuits), we have four machines and four qubits, which can give us at least 0 and at most 2 full machines because two pairs of qubits can be in two distinct machines. For $n=7,i=1$, we have $\lceil\frac{(n-\lfloor\frac{n}{2}\rfloor-i)}{2}\rceil=2$, and for $n=7,i=2$, we have $\lceil\frac{(n-\lfloor n/2\rfloor-i)}{2}\rceil=1$.

Induction Hypothesis: After the $\lfloor\frac{n}{2}\rfloor$-th front sub-circuit is executed in QFT_n, the minimum and maximum number of full machines in the quantum network are respectively zero and $\lceil\frac{(n-\lfloor\frac{n}{2}\rfloor-i)}{2}\rceil$, for $1\le i\le(n-2)-\lfloor\frac{n}{2}\rfloor$.

Inductive Step: We prove the lemma for QFT_{n+1}, and consider two cases. First, we let n be an even value, thus making $n+1$ an odd value. As such, $\lfloor\frac{n+1}{2}\rfloor=\frac{n}{2}$ and $1\le i\le((n+1)-2)-\lfloor\frac{n+1}{2}\rfloor$, which means $1\le i\le\frac{n}{2}-1$. After $\frac{n}{2}$ front sub-circuits are executed, $\frac{n}{2}$ qubits have been measured and there are $(n+1)-\frac{n}{2}=\frac{n}{2}+1$ qubits left. The number of machines is $m=\lceil\frac{n+1}{2}\rceil$, which is equal to $\frac{n+2}{2}=\frac{n}{2}+1$ since $n+1$ is odd. Now, for every front sub-circuit i (where $1\le i\le\frac{n}{2}-1$) that is executed from this point on, one additional qubit can be measured. Thus, if we have measured i additional qubits starting with the $\frac{n}{2}+1$ remaining qubits, then $\frac{n}{2}+1-i$ qubits remain, for $1\le i\le\frac{n}{2}-1$. Since we have $m=\frac{n}{2}+1$ machines and $\frac{n}{2}+1-i$ remaining qubits, there may be some qubit distribution where all machines are half-full; i.e., no machine is full. On the other hand, we can have $\lfloor\frac{(\frac{n}{2}+1-i)}{2}\rfloor$ pairs of qubits giving us at most $\lfloor\frac{(\frac{n}{2}+1-i)}{2}\rfloor$ full machines. This is exactly equal to $\lfloor\frac{(n+1-\lfloor\frac{n+1}{2}\rfloor-i)}{2}\rfloor$, which has a tight upper bound $\lceil\frac{(n+1-\lfloor\frac{n+1}{2}\rfloor-i)}{2}\rceil$. Using a similar reasoning, one can show that we have at least no full machine and at most $\lceil\frac{(n+1-\lfloor\frac{n+1}{2}\rfloor-i)}{2}\rceil$ full machines for the case where n is odd and $n+1$ becomes even.

Induction on i. We now prove the lemma by induction on i for a fixed odd n.

Base Case: Let $i=1$. Thus, after the execution of $(\lfloor\frac{n}{2}\rfloor+1)$-th front sub-circuit, we have $n-(\lfloor\frac{n}{2}\rfloor+1)$ remaining qubits (based on Lemma 4). Let n be an odd value, where $\lfloor\frac{n}{2}\rfloor=\frac{n-1}{2}$. Thus, $n-(\lfloor\frac{n}{2}\rfloor+1)=n-\frac{n-1}{2}-1=\frac{n-1}{2}$. Since the number of machines is $m=\lceil\frac{n}{2}\rceil=\frac{n+1}{2}$ (for odd values of n), we have more machines than qubits. As a result, the minimum number of full machines is zero where each qubit is in a distinct machine (i.e., all machines are either half-full or empty). The maximum number of full machines happens for a distribution where machines are filled up with qubits to the extent possible. That is, we have $\frac{n-1}{4}$ full machines. This matches with the statement of the lemma, where we have $\lceil\frac{(n-\lfloor\frac{n}{2}\rfloor-i)}{2}\rceil=\lceil\frac{n-\frac{n-1}{2}-1}{2}\rceil=\lceil\frac{n-1}{4}\rceil=\frac{n-1}{4}$ (because n is odd).

Now, let n be an even value, where $\lfloor\frac{n}{2}\rfloor=\frac{n}{2}$. Thus, we have $n-(\lfloor\frac{n}{2}\rfloor+1)=n-\frac{n}{2}-1=\frac{n-2}{2}$ remaining qubits after the execution of $(\lfloor n/2\rfloor+1)$-th front sub-circuit. Since the number of machines is $m=\lceil\frac{n}{2}\rceil=\frac{n}{2}$ (for even values of n), we have more machines than qubits. As a result, the minimum number of full machines is zero where each qubit is in a distinct machine (i.e., all machines are either half-full or empty). The maximum number of full machines happens for a distribution where machines are filled up with qubits to the extent possible. That is, we have $\frac{n-2}{4}$ full machines. This matches with the statement of the lemma, where we have $\lceil\frac{(n-\lfloor n/2\rfloor-i)}{2}\rceil=\lceil\frac{n-\frac{n}{2}-1}{2}\rceil=\lceil\frac{n-2}{4}\rceil=\frac{n-2}{4}$ (because n is even).

Induction Hypothesis: After the $\lfloor\frac{n}{2}\rfloor$-th front sub-circuit is executed in QFT_n, the minimum and maximum number of full machines in the quantum network are respectively zero and $\lceil\frac{(n-\lfloor\frac{n}{2}\rfloor-i)}{2}\rceil$, for $1\leq i\leq(n-2)-\lfloor\frac{n}{2}\rfloor$.

Inductive Step: We show that, after the $\lfloor\frac{n}{2}\rfloor+(i+1)$-th front sub-circuit is executed in QFT_n, the minimum and maximum number of full machines in the quantum network are respectively zero and $\lceil\frac{n-\lfloor\frac{n}{2}\rfloor-(i+1)}{2}\rceil$. Since the minimum number of full machines is zero after the $(\lfloor\frac{n}{2}\rfloor+i)$-th front sub-circuit is executed (based on induction hypothesis), executing one more front sub-circuit and measuring one more qubit will preserve the minimum at zero. As for the maximum, we have $n-(\lfloor\frac{n}{2}\rfloor+i+1)$ remaining qubits after the execution of the $\lfloor\frac{n}{2}\rfloor+i+1$-th front sub-circuit. Thus, there can be at most $\lceil\frac{n-(\lfloor\frac{n}{2}\rfloor+i+1)}{2}\rceil$ full machines, which is equal to $\lceil\frac{n-\lfloor\frac{n}{2}\rfloor-(i+1)}{2}\rceil$. □

Lemma 8. *If M_f is full in the first layer of the i-th front sub-circuit of QFT_n, where $1\leq i\leq n-2$, and there are $k\geq 1$ other full machines distinct from M_f, then the distribution of QFT_n over $m=\lceil\frac{n}{2}\rceil$ machines of capacity 2 requires at least $2k$ swaps and $((n+1-i)-2k)$ teleportation operations in the i-th front sub-circuit of QFT_n.*

Proof. Let the number of machines in the network be $m=\lceil\frac{n}{2}\rceil$, where there are k full machines and $m-k$ non-full machines. Moreover, we know that the i-th sub-circuit has $n-i$ layers (based on Lemma 2), in order containing the gates $R_2,\cdots,R_{n-i+1}$ (where $1\leq i\leq n-2$). The number of remaining qubits in the i-th front sub-circuit is $(n+1-i)$ (based on Lemma 4). Also, the number of qubits in full machines is $2k$, and the number of qubits in non-full machines is $(n+1-i)-2k$.

Case 1: q_t is in a half-full machine. Initially, one can teleport q_f to M_t, and locally execute the gate R_{i+1} whose input includes q_f and q_t. This can occur as many times as the number of qubits that are in half-full machines. The number of such qubits is equal to the total number of qubits at i-th front sub-circuit (i.e., $(n+1-i)$ based on Lemma 4) minus the total number of qubits in full machines (i.e., $2k$). Thus, we have $(n+1-i)-2k$ qubits that can be in half-full machines. That is, $(n+1-i)-2k$ teleportation operations occur. After these teleportations, q_t must be in a full machine M_t, and we are to execute some gate R_j, where $j=2+((n+1-i)-2k)$. (Note that, at the i-th front sub-circuit, we start from gate R_2 up to R_{n-i+1}.) Let q'_t be the neighboring qubit of q_t in M_t. Then, to execute R_j locally, either q'_f is swapped with q_t or q_f is swapped with q'_t. Wlog, let q'_f be swapped with q_t, which results in q_f and q_t ending up in the same (full) machine. A similar scenario occurs for the neighboring qubit q'_t, and another swap takes place. Since we have k full machines, we have $2k$ target qubits that are in full machines, where a swap must take place for each such qubit. Thus, we have to perform at least $2k$ swaps and $(n+1-i)-2k$ teleportations.

Case 2: q_t is in a full machine. Since M_f is in a full machine, a scenario similar to the previous case occurs until execution gets to a gate R_j where q_t is not in a full machine. At this point, q_f can be teleported to M_t. As long as q_t remains to be in a half-full machine for the subsequent gates, the same scenario occurs and one teleportation per gate will occur. Thus, we need at least $(n+1-i)-2k$ teleportations because $(n+1-i)-2k$ represents the number of qubits q_t in half-full machines. However, once execution reaches a gate where q_t is in a full machine, a swap is unavoidable to make that gate local (based on Lemma 3). Thus, for every gate whose q_t is in a full machine a swap must take place. Thus, we have to perform at least $2k$ swaps and $(n+1-i)-2k$ teleportations.

□

Lemma 9. *If M_f is half-full in the first layer of the i-th front sub-circuit of QFT_n, where $1 \leq i \leq n-2$, and there are $k \geq 1$ other full machines distinct from M_f, then the distribution of QFT_n over $m = \lceil \frac{n}{2} \rceil$ machines of capacity 2 requires at least $2k$ swaps and $((n+1-i)-2k)$ teleportation operations in the i-th front sub-circuit of QFT_n.*

Proof. Since both q_f and q_t are in half-full machines, we can teleport either one to the machine of the other qubit. This will result in M_f becoming a full machine. Thus, once we execute the first gate of the i-th front sub-circuit, q_f will end up in a full machine, and we are back to a scenario captured by Lemma 8, where execution starts with q_f being in a full machine. In this case, we have executed one gate and the number of remaining gates/layers is one less. However, this does not affect the number of full machines, and in turn has no impact on the minimum number of swaps and teleportations. □

Theorem 1. *To distribute QFT_n over a network of quantum machines with capacity 2 and a clique topology, we need at least $\frac{n^2-2n}{4}$ swaps and $\frac{n^2-2n-10}{4}$ teleportation operations, where n is even. If n is odd, the minimum cost of communication includes $\frac{n^2-1}{4}$ swaps and $\frac{5n^2-8n-17}{8}$ teleportations. Overall, the asymptotic lower bound of communication costs is $\Omega(n^2)$.*

Proof. We show this lemma by calculating the summations of the minimum number of swap and teleportations required across all front-sub-circuits of QFT$_n$. The minimum values depend on the minimum number of full machines in the i-th front sub-circuit, for $1 \leq i \leq n-2$. Based on Lemma 5, there are at least $m-i$ full machines for front sub-circuits from 1 to $\lceil n/2 \rceil$; i.e., $1 \leq i \leq \lceil n/2 \rceil$. Based on Lemmas 8 and 9, in the i-th front sub-circuit, the distribution of QFT$_n$ requires at least $2k$ swaps and $n+1-i-2k$ teleportations if there were k full machines, for $1 \leq i \leq n-2$. Thus, for $1 \leq i \leq \lceil n/2 \rceil$, we need at least $\Sigma_{i=1}^{\lceil n/2 \rceil} 2k = \Sigma_{i=1}^{\lceil n/2 \rceil} 2(m-i) = 2\Sigma_{i=1}^{\lceil n/2 \rceil} \lceil n/2 \rceil - i$ swaps and $\Sigma_{i=1}^{\lceil n/2 \rceil}(n+1-i-2(\lceil n/2 \rceil - i))$ teleportations. We calculate this summation for odd and even n as follows:

- If n is even, then $\lceil n/2 \rceil = n/2$. Thus, we have $2\Sigma_{i=1}^{n/2} n/2 - i = 2(n/2)^2 - 2\Sigma_{i=1}^{n/2} i = 2(n/2)^2 - 2[(n/2)(n/2+1)/2] = (n^2-2n)/4$ swaps and $\Sigma_{i=1}^{n/2}(n+1-i-2(n/2-i)) = \Sigma_{i=1}^{n/2}(i+1) = n/2 + \Sigma_{i=1}^{n/2} i = (n^2+6n)/8$ teleportations.
- When n is odd, we have $\lceil n/2 \rceil = (n+1)/2$. Thus, it follows that $2\Sigma_{i=1}^{(n+1)/2}(n+1)/2 - i = 2(n+1)^2/4 - 2\Sigma_{i=1}^{(n+1)/2} i = (n^2-1)/4$ swaps and $\Sigma_{i=1}^{(n+1)/2}(n+1-i-2((n+1)/2-i)) = \Sigma_{i=1}^{(n+1)/2} i = \frac{n^2+4n+3}{2}$ teleportations.

Since the minimum number of full machines after $\lfloor n/2 \rfloor$-th front sub-circuit is zero (based on Lemma 7), the distribution of QFT$_n$ would take only $\Sigma_{i=\lfloor n/2 \rfloor+1}^{(n-2)} n+1-i$ teleportations, for $\lfloor n/2 \rfloor < i \leq (n-2)$.

- If n is even, then $\lfloor \frac{n}{2} \rfloor = \frac{n}{2}$. As a result, we have $\Sigma_{i=\lfloor \frac{n}{2} \rfloor+1}^{(n-2)}((n+1)-i) = \Sigma_{i=\frac{n}{2}+1}^{(n-2)}((n+1)-i) = (n+1)(\frac{n-6}{2}) - \Sigma_{i=\frac{n}{2}+1}^{(n-2)} i = \frac{n^2-10n-20}{8}$, for $\lfloor n/2 \rfloor < i \leq (n-2)$ teleportations.

- If n is odd, then $\lfloor \frac{n}{2} \rfloor = \frac{n-1}{2}$. Thus, $\Sigma_{i=\frac{n-1}{2}+1}^{n-2}((n+1)-i) = \Sigma_{i=\frac{n+1}{2}}^{n-2}((n+1)-i) = (n+1)\frac{n-5}{2} - \Sigma_{i=\frac{n+1}{2}}^{n-2} i = \frac{n^2-4n-5}{2} - \frac{1}{2}(n\lceil \frac{n}{2} \rceil + (\lceil \frac{n}{2} \rceil + 1)^2)$, for $1 \leq i \leq n/2+3/2$ teleportations. Since n is odd, we have $\lceil \frac{n}{2} \rceil = \frac{n+1}{2}$, and as a result, $\frac{1}{2}(n\lceil \frac{n}{2} \rceil + (\lceil \frac{n}{2} \rceil + 1)^2) = \frac{3n^2+8n+9}{8}$. Therefore, $\Sigma_{i=\frac{n-1}{2}+1}^{n-2}((n+1)-i) = \frac{n^2-24n-29}{8}$.

Adding up the above number of teleportations to the number of teleportations for front sub-circuits from 1 to $\lceil n/2 \rceil$, we get the total number of $\frac{n^2-2n-10}{4}$ teleportations for even values of n and $\frac{5n^2-8n-17}{8}$ teleportations where n is odd. □

4 Lower Bounds for Networks of c-Qubit Machines for $c > 2$

This section studies the quantum communication costs when QFT_n is distributed on a clique network of quantum machines with capacity $c > 2$. Let $m = \lceil \frac{n}{c} \rceil$ denote the number of machines in the quantum network. We state the following lemma:

Lemma 10. *Let $n > 2$ be a multiple of c; i.e., initially all machines are full; $m = n/c$. After the i-th front sub-circuit is executed in QFT_n, where $1 \leq i \leq m$, the minimum number of full machines is $(m-i)$ and the maximum number of full machines is $m - \lceil \frac{i}{c} \rceil$ in the quantum network.*

Proof. We prove this lemma by an induction on i for a fixed n, and then an induction on n for a fixed i.
Base Case: We start the base case with $i = 1$. Since n is a multiple of c, all machines are initially full. After the first front sub-circuit is executed, one of the machines becomes non-full, thereby giving us a minimum of $m-1$ full machines. Moreover, since $c > 2$, $\lceil \frac{1}{c} \rceil = 1$. Thus, we get a maximum of $m-1$ full machines.

Induction Hypothesis: After the i-th front sub-circuit is executed in QFT_n, where $1 \leq i \leq m$, the minimum number of full machines is $(m-i)$ and the maximum number of full machines is $m - \lceil \frac{i}{c} \rceil$ in the quantum network.

Inductive Step: We show that, after the $(i+1)$-th front sub-circuit is executed in QFT_n, the minimum number of full machines is $(m-(i+1))$ and the maximum number of full machines is $m - \lceil \frac{i+1}{c} \rceil$, for $1 \leq i < m$. By induction hypothesis, we know that before executing the $(i+1)$-th front sub-circuit, the number of full machines is between $(m-i)$ and $m - \lceil \frac{i}{c} \rceil$. After the execution of the $(i+1)$-th front sub-circuit, one more qubit is measured. This qubit could be in a full machine, which would decrease the minimum of $(m-i)$ machines to $(m-i-1) = m-(i+1)$. Alternatively, the measured qubit could have been in a non-full machine, which will not change the number of full machines unless i is a multiple of c. In this case, $\lceil \frac{i+1}{c} \rceil$ will be one unit more than $\lceil \frac{i}{c} \rceil$. This will decrease the maximum from $m - \lceil \frac{i}{c} \rceil$ to $m - \lceil \frac{i+1}{c} \rceil$. Overall, the maximum number of full machines will be $m - \lceil \frac{i+1}{c} \rceil$ after the $(i+1)$-th front sub-circuit.

The induction proof on n for a fixed i has a similar outline to the proof of Lemmas 5 and 6, hence omitted. □

Lemma 11. *Let $n > 2$ be a value that is not a multiple of c; i.e., initially there are $m-1$ full machines where $m = \lceil n/c \rceil$. After the i-th front sub-circuit is executed in QFT_n, where $1 \leq i < m$, the minimum number of full machines in the quantum network is $m-1-i$ and the maximum number of full machines is $m - \lceil \frac{i+1}{c} \rceil$.*

Proof. We prove this lemma by an induction on i for a fixed n, and then an induction on n for a fixed i.
Base Case: We start the base case with $i = 1$. Since n is not a multiple of c, there are exactly $m-1$ full machines and a non-full machine initially. After the first front sub-circuit is executed, the top qubit q_f can be measured and two cases could occur. Either q_f was in the non-full machine, or it was in a full machine. In the former case, the number of full machines would not change, i.e., the minimum number of full machines is $m-1$ and the maximum number of full machines would be $m - \lceil \frac{2}{c} \rceil = m-1$ (since $c > 2$). However, in the latter case, the minimum number of full machines would be decreased to $m-2$. The maximum number of full machines would be the maximum over the two cases; i.e., $m-1$.

Induction Hypothesis: After the i-th front sub-circuit is executed in QFT_n, where $1 \leq i < m$, the minimum number of full machines is $(m-i-1)$ and the maximum number of full machines is $m - \lceil \frac{i+1}{c} \rceil$ in the quantum network.

Inductive Step: We show that, after the $(i+1)$-th front sub-circuit is executed in QFT_n, the minimum number of full machines is $(m-1-(i+1))$ and the maximum number of full machines is $m - \lceil \frac{i+2}{c} \rceil$, for $1 \leq i < m$. By induction hypothesis, we know that before executing the $(i+1)$-th front sub-circuit, the number of full machines is between $(m-1-i)$ and $m - \lceil \frac{i+1}{c} \rceil$. After the execution of the $(i+1)$-th front sub-circuit, one more qubit is measured. This qubit, denoted q_f, could be in a full machine, which would decrease the minimum of $(m-1-i)$ full machines to $(m-1-i-1) = m-1-(i+1)$. Alternatively, the measured qubit could have been in a non-full machine, which will not change the number of full machines unless $i+1$ is a multiple of c. In this case, $\lceil \frac{i+2}{c} \rceil$ will be one unit more than $\lceil \frac{i+1}{c} \rceil$. This will decrease the maximum from $m - \lceil \frac{i+1}{c} \rceil$ to $m - \lceil \frac{i+2}{c} \rceil$. Overall, the maximum number of full machine will be $m - \lceil \frac{i+2}{c} \rceil$ after the $(i+1)$-th front sub-circuit.

The induction proof on n for a fixed i has a similar outline to the proof of Lemmas 5 and 6, hence omitted. □

Corollary 2. *After the i-th front sub-circuits is executed in QFT_n, where $1 \leq i < m$, there can be no less than $m-i-1$ full machines in the quantum network. (Proof follows from Lemmas 10 and 11.)*

Lemma 12. *For front sub-circuits of QFT_n beyond m-th, the maximum number of full machines is $\lfloor \frac{n-m-i}{c} \rfloor$ and the minimum number of full machines is zero in each front sub-circuit i, for $1 \leq i \leq n-m-2$. (Recall that, $m = \lceil \frac{n}{c} \rceil$.)*

Proof. We prove this lemma by an induction on i starting from m-th front sub-circuit.
Base Case: Let $i = 1$; i.e., the execution is at the end of $m+1$-th sub-circuit. Thus, we have $m+1$ measured qubits and $n-m-1$ remaining qubits. Each machine will hold $\frac{n-m-1}{m}$ qubits on average. We show that $\frac{n-m-1}{m} < c$, representing a scenario where qubits are distributed evenly and no machine is full. Observe that, $\frac{n-m-1}{m} = \frac{n}{m} - 1 - \frac{1}{m}$. If n is a multiple of c, then $\frac{n}{m} = c$, and as a result, $\frac{n}{m} - 1 - \frac{1}{m} = c - 1 - \frac{1}{m} < c$. If n is not a multiple of c, then initially $m-1$ machines are full and one machine is non-full with less than c qubits; i.e., $n = c(m-1) + r$, where $r < c$. Now, we show that $\frac{n-m-1}{m} < c$. Observe that, $\frac{n-m-1}{m} = \frac{cm-c+r-m-1}{m} = c - \frac{c}{m} + \frac{r}{m} - 1 - \frac{1}{m}$. Since $r < c$, the value of $-\frac{c}{m} + \frac{r}{m}$ is negative. Thus, the expression $c - \frac{c}{m} + \frac{r}{m} - 1 - \frac{1}{m}$ is strictly less than c. Therefore, if the remaining $n-m-1$ qubits are evenly distributed, then there are no full machines in the network. The maximum occurs when we fill up the machines one after another using the remaining $n-m-1$ qubits. That is, the maximum number of full machines is $\lfloor \frac{n-m-1}{c} \rfloor$.

Induction Hypothesis: For front sub-circuits of QFT_n beyond m-th, the maximum number of full machines is $\lfloor \frac{n-m-i}{c} \rfloor$ and the minimum number of full machines is zero in each front sub-circuit i, for $1 \leq i \leq n-m-2$.

Inductive Step: We show that, if for some $(m+i)$-th front sub-circuit, where $1 < i < n-m-2$, the maximum number of full machines is $\lfloor \frac{n-m-i}{c} \rfloor$ and the minimum number of full machines is zero, then for $(m+i+1)$-th front sub-circuit the maximum number of full machines is $\lfloor \frac{n-m-i-1}{c} \rfloor$ and the minimum number of full machines is zero. Let the execution be at the end of the $(i+1)$-th sub-circuit. Thus, we can measure the top qubit q_f and remain with $n-m-(i+1)$ qubits. If we allocate these qubits in a way that machines are filled up one after another, then we can fill up $\lfloor \frac{n-m-i-1}{c} \rfloor$ machines of capacity c, thereby getting maximum $\lfloor \frac{n-m-i-1}{c} \rfloor$ full machines. Based on the base case and the fact that $i > 1$, even distribution of $n-m-i-1$ qubits across m machines of capacity c would leave no full machines, giving us the minimum of zero full machines. □

Lemma 13. *Let M_f be full. In the i-th front sub-circuit of QFT_n, where $1 \leq i \leq n-2$, if there are k other full machines distinct from M_f (where $0 \leq k \leq m$), then the distribution of QFT_n over $m = \lceil \frac{n}{c} \rceil$ machines of capacity c in a clique network requires at least ck swaps and at least $\lceil \frac{(n+1-i)-ck}{c-1} \rceil$ teleportation operations.*

Proof. Case 1: q_t is in a non-full machine. Initially, one can teleport q_f to M_t, and locally execute the gate whose input includes q_f and q_t. (Note that, if q_t is in M_f, the current gate can be executed locally.)

The maximum number of teleportations that one can perform is equal to the number of qubits that are in non-full machines (where q_t of the next gate is always in a different machine than the current machine). The minimum number of teleportations is equal to the number of non-full and non-empty machines, which is equal to the number of remaining qubits in non-full machines divided by $c-1$ (because each non-full machine can have at most $c-1$ qubits). The number of qubits in non-full machines is equal to the total number of qubits at the i-th front sub-circuit (i.e., $(n+1-i)$ based on Lemma 4) minus the total number of qubits in full machines (i.e., ck). Note that, M_f is initially full and after the first gate is executed q_f might be teleported to a non-full machine. Thus, we have $(n+1-i)-ck$ qubits in non-full machines. The best case occurs when q_f is teleported to a machine and all the target qubits of the next $c-1$ gates are in M_f too. Thus, the minimum number of teleportations in this case is equal to $\lceil \frac{(n+1-i)-ck}{c-1} \rceil$. We consider the ceiling because the last non-full machine may have less than $c-1$ qubits, but still needs an additional teleportation for q_f. The maximum number of teleportations occurs when q_f should be moved to a different machine as the next gate is to be executed; i.e., $(n+1-i)-ck$ teleportation operations. (There is no need to teleport the qubits that are in M_f.) After these teleportations, q_t may end up in a full machine M_t, and we should execute some gate R_j, where $j = 2 + ((n+1-i)-ck)$. Let q'_t be a neighboring qubit of q_t in M_t. Then, to execute R_j locally, either some neighboring qubit of q_f, denoted q'_f, is swapped with q_t or q_f is swapped with q'_t. Wlog, let q'_f be swapped with q_t, which results in q_f and q_t ending up in the same machine. (Alternatively, one could argue that we could teleport q_f to another machine M_e that contains at most $c-2$ qubits and then teleport q_t to M_e too. This would also take two teleportations, which costs the same as a swap operation.) Next, the gate R_{j+1} should be executed locally whose inputs include q_f and some other qubit q_t. A similar scenario occurs and another swap must take place. Thus, for each full machine (distinct from M_f) at least c swaps take place, which results in a total ck swaps in the i-th front sub-circuit.
Case 2: q_t is in a full machine. Since q_t is in a full machine, a sequence of swaps occurs until execution gets to a gate R_j whose q_t is in a non-full machine. At this point, q_f can be teleported to M_t. As long as q_t remains in a non-full machine for the subsequent gates, the same scenario as Case 1 occurs in terms of the number of teleportations. However, once execution reaches a gate where q_t is in a full machine, a swap is unavoidable to make that gate local (see Lemma 3). Thus, for every gate whose q_t is in a full machine a swap must take place. Since we have k full machines in addition to M_f, we have ck target qubits that are in full machines, where a swap must take place for each such qubit. □

Lemma 14. *Let M_f be non-full. In the i-th front sub-circuit of QFT_n, where $1 \leq i \leq n-2$, if there are k full machines (where $0 \leq k \leq m$), then the distribution of QFT_n over $m = \lceil \frac{n}{c} \rceil$ machines of capacity c in a clique network requires at least $ck - c$ swaps and at least $\lceil \frac{(n+1-i)-ck}{c-1} \rceil$ teleportation operations.*

Proof. Case 1: q_t is in a non-full machine. Initially, one can teleport q_f to M_t so the current gate can be locally executed. Similar to the proof of Lemma 13, the minimum number of teleportations is equal to the number of non-full and non-empty machines. To determine the number of such machines we calculate the number of qubits in non-full machines and divide it by $c-1$. The number of qubits in non-full machines is $(n+1-i)-ck$. Thus, the minimum number of teleportations in this case is $\lceil \frac{(n+1-i)-ck}{c-1} \rceil$. Once the execution reaches a gate whose q_t is in a full machine, target qubits can be teleported to M_f until M_f becomes full, thereby decreasing the number of full machines distinct from M_f to $k-1$. When that happens, subsequent gates whose target qubit is in a full machine can only be localized through a swap operation because we reach a configuration similar to that of Lemma 13 where M_f is full and there are $k-1$ other distinct full machines. Thus, in the i-th front sub-circuit of QFT$_n$, the minimum number of swaps is $c(k-1)$.
Case 2: q_t is in a full machine. This case is similar to the analysis of Case 1, where we reach a gate whose q_t is in a full machine. □

Corollary 3. *Consider a clique network of quantum machines of capacity c qubits, where $2 < c < n$ and n denotes the number of qubits. To distributed the i-th front sub-circuit of QFT_n, for $1 \leq i \leq n-2$, over the quantum network, we need a minimum number of ck swaps and $\lceil \frac{(n+1-i)-ck}{c-1} \rceil$ teleportation operations, where k denotes the number of full machines and $0 \leq k \leq m$.*

Proof. Proof follows from Lemmas 13 and 14. We use the larger lower bound on the number of swaps because we would like to have a tight bound regardless of M_f being full or not. ☐

Theorem 2. *Let n be a multiple of c; i.e., $m = \frac{n}{c}$. To distribute QFT_n over a network of quantum machines with capacity $2 < c < n$ and a clique topology, we need at least $\frac{n(n-c)}{2c}$ swaps and $\frac{n^2}{2c} + \frac{n}{c(c-1)} - \frac{3}{c-1}$ teleportation operations. Overall, the asymptotic lower bound of communication costs is $\Omega(n^2)$, assuming c is a constant.*

Proof. The minimum number of full machines in the i-th front sub-circuit of QFT_n is $k = m - i$ for $i \leq m$. When $i > m$, then the minimum number of full machines is zero because we can have one measured qubit per machine, which would make all machines non-full. Consider at least ck swaps and at least $\lceil \frac{((n+1-i)-ck)}{(c-1)} \rceil$ teleportation operations for the i-th front sub-circuit, where $1 \leq i \leq n - m$. Next, we calculate the number of swaps and teleportations separately.

- *Number of swap operations*: $\Sigma_{i=1}^{m} ck = \Sigma_{i=1}^{m} c(m-i) = c\Sigma_{i=1}^{m}(m-i) = c(m^2 - \Sigma_{i=1}^{m} i) = c(m^2 - m(m+1)/2)) = cm(m-1)/2$. Since n is a multiple of c, $m = \lceil \frac{n}{c} \rceil = \frac{n}{c}$ and the number of swaps is: $\frac{n(n-c)}{2c}$. *Observe that, when $c = 2$, we get $\frac{n^2-2n}{4}$, which is the value we computed in the proof of Theorem* 1.
- *Number of teleportation operations*: $\Sigma_{i=1}^{m} \lceil \frac{(n+1-i)-ck)}{(c-1)} \rceil$ is greater than or equal to $\Sigma_{i=1}^{m} \frac{(n+1-i)-ck)}{(c-1)}$, where $k = (m - i)$. Since our objective is to find an optimal lower bound, we compute the minimum number of teleportations based on $\Sigma_{i=1}^{m} \frac{(n+1-i)-ck)}{(c-1)}$, which is equal to $\frac{(c+1+2n)m-(c+1)m^2}{2(c-1)}$. Since n is a multiple of c, we have $m = \lceil \frac{n}{c} \rceil = \frac{n}{c}$, which leads to $\frac{(c+1+2n)m-(c+1)m^2}{2(c-1)} = \frac{n}{2(c-1)} + \frac{n}{2c(c-1)} + \frac{n^2}{2c^2}$. For $c = 2$, we have $\frac{n^2+6n}{8}$, which *matches with the value computed in the proof of Theorem 1 for even values of n.*

Since the minimum number of full machines after m-th front sub-circuit is zero (based on Lemma 12), the distribution of QFT_n would take only $\Sigma_{i=m+1}^{(n-2)} \lceil \frac{(n+1-i)}{c-1} \rceil$ teleportations in the best case. We lift the ceiling and calculate the minimum number of teleportations based on $\Sigma_{i=m+1}^{(n-2)} \frac{(n+1-i)}{c-1}$ because we would like to find the best we can achieve.

$\Sigma_{i=m+1}^{(n-2)} \frac{(n+1-i)}{c-1} = \frac{1}{c-1}(\Sigma_{i=m+1}^{(n-2)}(n+1-i)) = \frac{(n-m-3)(n+1)}{c-1} - \frac{1}{c-1}\Sigma_{i=m+1}^{(n-2)} i = \frac{(n-m-3)(n+1)}{c-1} - \frac{(n-m-3)(m+1)}{c-1} - \Sigma_{i=1}^{(n-m-3)} i = \frac{(n-m-3)(n+1)}{c-1} - \frac{(n-m-3)(m+1)}{c-1} - \frac{(n-m-3)(n-m-2)}{2(c-1)} = \frac{(n-m-3)(n-m+2)}{2(c-1)}$.

When n is a multiple of c, we have $m = \frac{n}{c}$, which in turn results in $\frac{(n-m-3)(n-m+2)}{2(c-1)} = \frac{n^2(c-1)}{2c^2} - \frac{n}{2c} - \frac{3}{c-1}$. The overall minimum number of teleportations is $\frac{n}{2(c-1)} + \frac{n}{2c(c-1)} + \frac{n^2}{2c^2} + \frac{n^2(c-1)}{2c^2} - \frac{n}{2c} - \frac{3}{c-1} = \frac{n^2}{2c} + \frac{n}{c(c-1)} - \frac{3}{c-1}$ ☐

Theorem 3. *Assume that n is not a multiple of c, where $m = \lceil \frac{n}{c} \rceil$, and $n = cq + r$, for some integers $0 < q < n$ and $r < c$. To distribute QFT_n over a network of quantum machines with capacity $2 < c < n$ and a clique topology, we need at least $\frac{(n-r)^2+c(n-r)}{2c}$ swaps and $\frac{n^2}{2c} + \frac{n(r-c)}{c^2} + \frac{r^2(c+2)}{2c^2(c-1)} - \frac{r(c+4)}{2c(c-1)} - \frac{2}{(c-1)}$ teleportation operations. Overall, the asymptotic lower bound of communication costs is $\Omega(n^2)$, assuming c is a constant.*

Proof. The minimum number of full machines in the i-th front sub-circuit of QFT_n is $k = m - i$ for $i \leq m$. When $i > m$, then the minimum number of full machines is zero because we can have one measured qubit per machine, which would make all machines non-full. Consider at least ck swaps and at least $\lceil((n+1-i)-ck)/(c-1)\rceil$ teleportation operations for the i-th front sub-circuit, where $1 \leq i \leq n - m$. Next, we calculate the number of swaps and teleportations separately.

- *Number of swap operations*: From the proof of Theorem 2, we have the number of swaps as $\Sigma_{i=1}^{m} ck = \frac{cm(m-1)}{2}$. Let $n = cq + r$, where $r < c$. Thus, $m = \lceil \frac{n}{c} \rceil = q + 1$. As a result, we have $\Sigma_{i=1}^{m} ck = \frac{cm(m-1)}{2} = \frac{cq(q+1)}{2}$. Since $n = cq + r$, we have $\Sigma_{i=1}^{m} ck = \frac{(n-r)^2 + c(n-r)}{2c}$. For odd values of n and $c = 2$, we have $r = 1$, which means $\Sigma_{i=1}^{m} ck = \frac{n^2-1}{4}$, which is *the same value obtained for the number of swaps (when n is odd) in the proof of Theorem* 1.

- *Number of teleportation operations*: $\Sigma_{i=1}^{m} \lceil \frac{(n+1-i)-ck)}{(c-1)} \rceil$ is greater than or equal to $\Sigma_{i=1}^{m} \frac{(n+1-i)-ck)}{(c-1)}$, where $k = (m - i)$. Since our objective is to find an optimal lower bound, we compute the minimum number of teleportations based on $\Sigma_{i=1}^{m} \frac{(n+1-i)-ck)}{(c-1)}$, which is equal to $\frac{(c+1+2n)m-(c+1)m^2}{2(c-1)}$. Let $n = cq + r$, where $r < c$. Thus, $m = \lceil \frac{n}{c} \rceil = q + 1$. As a result, we have $\frac{(c+1+2n)m-(c+1)m^2}{2(c-1)} = \frac{n^2}{2c^2} + \frac{n}{2c} + \frac{(c+1)r^2}{2c^2(c-1)} - \frac{(c+1)r}{2c(c-1)}$. For odd values of n and $c = 2$, we have $\frac{n^2+2n-3}{8}$, which is smaller than what we calculated in Theorem 1 (i.e., $\frac{n^2+4n+3}{2}$). This is due to considering $\Sigma_{i=1}^{m} \frac{(n+1-i)-ck)}{(c-1)}$ instead of $\Sigma_{i=1}^{m} \lceil \frac{(n+1-i)-ck)}{(c-1)} \rceil$.

Since the minimum number of full machines after m-th front sub-circuit is zero (based on Lemma 12), the distribution of QFT$_n$ would take only $\Sigma_{i=m+1}^{(n-2)} \lceil \frac{(n+1-i)}{c-1} \rceil$ teleportations in the best case. We lift the ceiling and calculate the minimum number of teleportations based on $\Sigma_{i=m+1}^{(n-2)} \frac{(n+1-i)}{c-1}$ because we would like to find an optimal lower bound.

$$\Sigma_{i=m+1}^{(n-2)} \frac{(n+1-i)}{c-1} = \frac{1}{c-1}(\Sigma_{i=m+1}^{(n-2)}(n + 1 - i)) = \frac{(n-m-3)(n+1)}{c-1} - \frac{1}{c-1}\Sigma_{i=m+1}^{(n-2)} i = \frac{(n-m-3)(n+1)}{c-1} - \frac{(n-m-3)(m+1)}{c-1} - \Sigma_{i=1}^{(n-m-3)} i = \frac{(n-m-3)(n+1)}{c-1} - \frac{(n-m-3)(m+1)}{c-1} - \frac{(n-m-3)(n-m-2)}{2(c-1)} = \frac{(n-m-3)(n-m+2)}{2(c-1)}$$

Since $m = q + 1$ and $q = \frac{n-r}{c}$, we have $\frac{(n-m-3)(n-m+2)}{2(c-1)} = \frac{n^2(c-1)}{2c^2} + \frac{(2r-3c)n}{2c^2} + \frac{r^2-3cr-4c^2}{2c^2(c-1)}$. Adding this to the minimum number of teleportations we obtain for $1 \leq i \leq m$; i.e., $\frac{n^2}{2c^2} + \frac{n}{2c} + \frac{(c+1)r^2}{2c^2(c-1)} - \frac{(c+1)r}{2c(c-1)}$, we get $\frac{n^2}{2c} + \frac{n(r-c)}{c^2} + \frac{r^2(c+2)}{2c^2(c-1)} - \frac{r(c+4)}{2c(c-1)} - \frac{2}{(c-1)}$ as the overall minimum number of teleportations. □

Corollary 4. *If $c = \frac{n}{t}$ where $t > 1$ is a constant independent of the number of machines, then the asymptotic lower bound of the communication costs is linear.*

Corollary 4 implies that as the number of qubits n increases, we need to scale up the machine capacity with constant ratio of $t = \frac{n}{c}$ in order to achieve linear asymptotic cost.

5 Related Work

There is a rich body of work [16, 19, 2, 13] on efficient implementation of QFT in hardware. Such methods are mostly hardware-dependent (e.g., available gate set, proximity noise of qubits) and are constrained by topological limits of hardware devices (e.g., linear nearest neighbor topology) which may not hold in quantum networks. Additionally, many researchers focus on minimizing the number of remote operations (e.g., TeleData/TeleGate) during distribution of quantum circuits in general, without a focus on lower bound. Such methods can be classified into graph-theoretic, heuristic-based and distribution compilers. As an example of graph-theoretic techniques, Andres-Martinez and Heunen [1] reduce the minimization problem to the problem of hypergraph partitioning where the number of cuts in the partitioned graph must be minimized. While this approach is efficient for some circuits, the hypergraph partitioning is by itself a hard problem. Davarzani *et al.* [7] create a bipartitie graph out of a quantum circuit where the two sets of vertices include the qubits and the gates, and solving TMP amounts to partitioning the set of qubits while minimizing the number of teleportations. Daei *et al.* [6] model a quantum circuit as a weighted undirected graph and utilize Kernighan–Lin's [12] algorithm to find the minimum-weight cut.

Heuristic-based approaches improve the efficiency of minimization. For example, Nikahd *et al.* [18] present a window-based partitioning method where they consider a window of length w that slides from leftmost layer to right and creates sub-circuits. They formulate the problem as an Integer Linear Program (ILP), which they solve using the CPLEX ILP solver. Ranjani and Gupta [21] present two algorithms: a local-best algorithms and a zero-stitching algorithm. Their local-best algorithm is a greedy approach in nature while their zero-stitching method is a dynamic programming approach to partition the circuit into sub-circuits that can be executed without any teleportations, and then stitching them together. To solve TMP, Zomorodi-Moghadam *et al.* [28] explore all possible configurations of executing every global gate in either one of two machines, which has an exponential cost.

Most recent DQC compilers focus on identifying communication patterns in quantum circuit as well as implementing minimized teleportation plans on quantum networks. For example, Wu *et al.* [25] observe that many remote two-qubit gates can be executed using one or two quantum communications, called *burst communication*. In another work, Wu *et al.* [24] develop a compiler based on the notion of *collective communication blocks*, where each block is a set of global gates whose pattern of qubit communication forms a connected graph over multiple network nodes. Ferrari *et al.* [10] discuss the challenges of developing compilers for DQC, and then present an upper bound complexity for the compilation process.

6 Conclusion and Future Work

This paper investigated lower bounds on the complexity of quantum communication for the distribution of Quantum Fourier Transform (QFT) in clique networks. We first discovered the relation between lower bound and the number of full machines, and presented necessary and conditions for unavoidable swaps. Then, we identified lower and upper bound on the number of full machines. We used this result to provide exact and asymptotic lower bound on the communication costs of distributing QFT on clique networks. Considering the existing quadratic upper bound in related work, our lower bound result implies that the existing upper bound is actually tight when machine capacity is two. This is a significant theoretical boundary for developers of distribution compilers. We also showed that if machine capacity $c > 2$ is a fixed fraction of n, then the lower bound becomes linear. We are currently extending these results for topologies other than clique. Moreover, we plan to study the communication complexity of distributing other building blocks of quantum algorithms (e.g., quantum phase estimation) as well as Variational Quantum Algorithms.

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