Qualifying Exam for Mathematical Statistics, Fall 2020

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Directions: (1) You need a calculator for some questions. (2) **Answer ALL problems.** (3) You are only required to answer what is asked so be brief but be precise.(4) Partial credit will be given to incomplete answers so show relevant work. (5) You may use one part of the problem to solve the other part, regardless whether you are able to solve the part you use. (6) **DO NOT** write your names on any place of your answer paper. (7) **DO NOT** write on the back of your answer paper. (8) Start each question on a **NEW** page.

Problem 1. 19 children are at a birthday party. Each child is asked if they would like to have a red balloon. Each child is also asked if they would like to have a yellow balloon. Therefore some children have a red balloon, some children have a yellow balloon, some children have both, and some children have neither. 7 children have a red balloon. 3 children have zero balloons. 5 children have two balloons. Draw a Venn Diagram with this information and answer the following questions:

(1) If a child at the party is selected at random, what is the probability they have exactly one balloon?

(2) If a child at the party is selected at random, what is the probability they have a red balloon OR they don't have a yellow balloon?

(3) If a child at the party is selected at random, what is the probability they have a yellow balloon given they have a red balloon?

Problem 2. A basket contains 5 orange, 6 yellow, and 7 white balls. Identify each of the following random variables as Binomial, Geometric, Negative Binomial, or Hypergeometric. Identify values of the parameters and possible values of each random variable.

(1) Draw 5 balls all at once at random **without** replacement. A = # of white balls.

(2) Draw 5 balls one at a time at random **with** replacement. B = # of white balls.

(3) Draw balls from the basket one at a time at random **with** replacement until you see the first yellow ball. C = # of draws needed.

(4) Draw balls from the basket one at a time at random **with** replacement until you see the third yellow ball. D = # of draws needed.

(5) Draw balls from the basket one at a time at random **with** replacement until you see a ball that isn't orange. E = # of draws needed.

(6) Draw 10 balls all at once at random **without** replacement. F = # of balls that aren't orange.

Problem 3. A continuous random variable has the following pdf:

$$f_Y(y) = \begin{cases} \frac{2}{3}y - \frac{2}{3} & 1 < y \le 2 \\ -\frac{1}{3}y + \frac{4}{3} & 2 < y < 4 \\ 0 & otherwise \end{cases}$$

(1) Set up (but not to evaluate) an integration that would find $E\left(\frac{1}{Y}\right)$.

(2) Find the cumulative distribution function (cdf) of Y

Problem 4. Two continuous random variables have the following joint pdf:

$$f(y_1, y_2) = \begin{cases} \frac{1}{162}(y_1 \cdot y_2) & 0 < y_1 < 6, \ 0 < y_2 < 6, \ \ y_1 > y_2 \\ 0 & otherwise \end{cases}$$

(1) Draw a picture of the support of $f(y_1, y_2)$. Set up (but do not evaluate) two double integrals that would find $P(Y_1 + Y_2 < 6)$. **Hint**: one of the two double integrals will have to be split into two pieces.

(2) Find the marginal pdf of Y_1.

(3) **Given**: the marginal pdf of Y_2 is:

$$f_2(y_2) = \begin{cases} \frac{y_2}{9} - \frac{y_2^3}{324} & 0 < y_2 < 6 \\ 0 & otherwise \end{cases}$$

Find the conditional pdf of Y_1 given $Y_2 = y_2$.

(4) Use your answer to the previous question to find $P(Y_1 > 3 | Y_2 = 2)$. You may use your calculator.

<u>Problem 5.</u> Mrs. Hopkins enjoys watching hummingbirds approach her hummingbird feeder on a summer afternoon. X = the # of hummingbirds that approach the feeder per minute. Assume X is a Poisson random variable with a mean of 4.7 hummingbirds per minute.

(1) What is the probability exactly 3 hummingbirds will approach her feeder within the next minute?
(2) What is the probability more than 2 hummingbirds will approach her feeder within the next minute?
(3) Since the # of hummingbirds per minute is a Poisson random variable, the waiting time until the next hummingbird appearance is a continuous Exponential random variable. What is the probability the waiting time for the next hummingbird appearance will be between 0.4 minutes and 0.7 minutes? You may use your calculator.

<u>Problem 6.</u> Let

Y_1 = height of a randomly selected 16-year-old boy (cm) and $Y_1 \sim N(\mu_1 = 173, \sigma_1 = 4)$

Y_2 = height of a randomly selected 16-year-old girl (cm) and $Y_2 \sim N(\mu_2 = 170, \sigma_2 = 3)$

Y_1 and Y_2 are independent.

(1) Find the probability one randomly selected 16-year-old boy is taller than 178.8 cm.
(2) Four 16-year-old girls are randomly selected. Find the probability the sample mean among their heights is less than 168.8 cm.
(3) If one 16-year-old boy and one 16-year-old girl is randomly selected, what is the probability the boy is taller than the girl?

<u>Problem 7.</u> Let $Y_1, \cdots, Y_5$ denote a random sample of size 5 from a Poisson distribution with probability mass function $p(y|\theta) = \exp(-\theta)\frac{\theta^y}{y!}, \theta > 0, y = 0,1,2,\cdots$. We are interested in testing $H_0: \theta \leq 1$ versus $H_1: \theta > 1$.

(1) The test statistic $T_1 = \sum_{i=1}^{5} I(Y_i \geq 2)$ is used, where $I(\cdot)$ is the indicator function. The null hypothesis is rejected when $T_1 \geq 4$. Find the size of this test.
(2) Construct the uniform most power (UMP) test. Find the exact rejection region so that the size of this test is same as the size of test in 1.

(3) If the true value of θ is 2, calculate the power of the test in (1) and the power of the test in (2).

(4) Find a one-sided confidence interval of θ based on the test in (2).

(5) If we observed $Y_1 = 1, Y_2 = 4, Y_3 = 3, Y_4 = 2$, and $Y_5 = 1$, calculate the p-value for the test in (1) and the test in (2).

Problem 8. Let $Y_1, \cdots, Y_n$ denote a random sample of size n from the following probability density function

$$f(y) = \theta y^{\theta-1}, 0 < y < 1, \theta > 0$$

(1) Find an estimator of θ based on the method of moments.

(2) Show that the estimator from (1) is the consistent estimator of θ.

(3) Find the maximum likelihood estimator of θ.

(4) Show that the estimator from (3) is the consistent estimator of θ.

(5) Without calculate the bias mean squared error of the estimator from (1) and (3), which estimator is asymptotically better? Why?

Standard Normal Distribution: Areas to the right of $z = .00, .01, .02, \ldots, 3.99$.

$$\Pr(Z > z) = \int_z^\infty \frac{1}{\sqrt{2\pi}}\ e^{-t^2/2}\ dt$$

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.5000	.4960	.4920	.4880	.4840	.4801	.4761	.4721	.4681	.4641
0.1	.4602	.4562	.4522	.4483	.4443	.4404	.4364	.4325	.4286	.4247
0.2	.4207	.4168	.4129	.4090	.4052	.4013	.3974	.3936	.3897	.3859
0.3	.3821	.3783	.3745	.3707	.3669	.3632	.3594	.3557	.3520	.3483
0.4	.3446	.3409	.3372	.3336	.3300	.3264	.3228	.3192	.3156	.3121
0.5	.3085	.3050	.3015	.2981	.2946	.2912	.2877	.2843	.2810	.2776
0.6	.2743	.2709	.2676	.2643	.2611	.2578	.2546	.2514	.2483	.2451
0.7	.2420	.2389	.2358	.2327	.2296	.2266	.2236	.2206	.2177	.2148
0.8	.2119	.2090	.2061	.2033	.2005	.1977	.1949	.1922	.1894	.1867
0.9	.1841	.1814	.1788	.1762	.1736	.1711	.1685	.1660	.1635	.1611
1.0	.1587	.1562	.1539	.1515	.1492	.1469	.1446	.1423	.1401	.1379
1.1	.1357	.1335	.1314	.1292	.1271	.1251	.1230	.1210	.1190	.1170
1.2	.1151	.1131	.1112	.1093	.1075	.1056	.1038	.1020	.1003	.0985
1.3	.0968	.0951	.0934	.0918	.0901	.0885	.0869	.0853	.0838	.0823
1.4	.0808	.0793	.0778	.0764	.0749	.0735	.0721	.0708	.0694	.0681
1.5	.0668	.0655	.0643	.0630	.0618	.0606	.0594	.0582	.0571	.0559
1.6	.0548	.0537	.0526	.0516	.0505	.0495	.0485	.0475	.0465	.0455
1.7	.0446	.0436	.0427	.0418	.0409	.0401	.0392	.0384	.0375	.0367
1.8	.0359	.0351	.0344	.0336	.0329	.0322	.0314	.0307	.0301	.0294
1.9	.0287	.0281	.0274	.0268	.0262	.0256	.0250	.0244	.0239	.0233
2.0	.0228	.0222	.0217	.0212	.0207	.0202	.0197	.0192	.0188	.0183
2.1	.0179	.0174	.0170	.0166	.0162	.0158	.0154	.0150	.0146	.0143
2.2	.0139	.0136	.0132	.0129	.0125	.0122	.0119	.0116	.0113	.0110
2.3	.0107	.0104	.0102	.0099	.0096	.0094	.0091	.0089	.0087	.0084
2.4	.0082	.0080	.0078	.0075	.0073	.0071	.0069	.0068	.0066	.0064
2.5	.0062	.0060	.0059	.0057	.0055	.0054	.0052	.0051	.0049	.0048
2.6	.0047	.0045	.0044	.0043	.0041	.0040	.0039	.0038	.0037	.0036
2.7	.0035	.0034	.0033	.0032	.0031	.0030	.0029	.0028	.0027	.0026
2.8	.0026	.0025	.0024	.0023	.0023	.0022	.0021	.0021	.0020	.0019
2.9	.0019	.0018	.0018	.0017	.0016	.0016	.0015	.0015	.0014	.0014
3.0	.0013	.0013	.0013	.0012	.0012	.0011	.0011	.0011	.0010	.0010
3.1	.0010	.0009	.0009	.0009	.0008	.0008	.0008	.0008	.0007	.0007
3.2	.0007	.0007	.0006	.0006	.0006	.0006	.0006	.0005	.0005	.0005
3.3	.0005	.0005	.0005	.0004	.0004	.0004	.0004	.0004	.0004	.0003
3.4	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0002
3.5	.0002	.0002	.0002	.0002	.0002	.0002	.0002	.0002	.0002	.0002
3.6	.0002	.0002	.0001	.0001	.0001	.0001	.0001	.0001	.0001	.0001
3.7	.0001	.0001	.0001	.0001	.0001	.0001	.0001	.0001	.0001	.0001
3.8	.0001	.0001	.0001	.0001	.0001	.0001	.0001	.0001	.0001	.0001
3.9	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000

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